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Lower bounds on multiple distinct sums sets

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Abstract

Improved lower bounds on multiple distinct sums sets are given. Lower bounds for the more general case of multiple difference set of a distinct sum set are considered.

Keywords: Distinct sums sets; Difference sets; Lower bounds.

1. Introduction

A $(2,J)$ distinct sum (DS) set (also known as a B_2-sequence) is a set $A = \{a_1, a_2, \ldots, a_J\}$ of positive integers such that all the $J(J+1)/2$ sums $a_j + a_{j'}$ where $1 \leqslant j' \geqslant j \leqslant J$ are distinct. It is easy to see that the sums are distinct if and only if the $J(J-1)/2$ differences $a_j - a_{j'}$ where $1 \leqslant j' < j \leqslant J$ are distinct. Much effort has been made to find good bound on the size of the maximal element of a $(2,J)$-DS. All lower bounds are found using the fact that the differences are distinct, see e.g. [9]. A number of papers consider the generalization where one have more than one DS at the same time, or where each difference can occur more than once, but only a limited number of times (i.e. $\leqslant \mu$ times for some fixed μ), or both; see e.g. [5].

Similarly, a (h,J)-DS is a set $A = \{a_1, a_2, \ldots, a_J\}$ such that all sums of exactly h elements are distinct. Such a sum can be represented by $(x_1, x_2, \ldots, x_J)$, where x_j is the number of times a_j appears in the sum. We use this notation below. To get good bounds one tries to do something similar to what was done for $h = 2$. Suppose that $h = 2q$, an even number. The sums can be written as $S_1 + S_2$ where S_1 and S_2 are sums of q elements from A. Again, one can consider the corresponding differences $S_1 - S_2$ and the methods for $h = 2$ can be modified to work also in this case. However, there are a number of extra complications, e.g. the splitting of a sum into the two sums is

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not unique. For some bounds obtained this way, see [3]. Again one can generalize to consider more than one set and also allow a difference to appear a limited number of times. This is the situation we will consider in this paper, and a main objective is to generalize the results of [5] to get improved lower bounds.

In the next section we give the basic notations, quote some known results, and describe precisely our problem. In the following sections we give the new results.

2. Basic notations and results

Let q and J be positive integers. Define

$$C(q,J) = \left\{ \bar{x} \,|\, x_j \text{ non-negative integers and } \sum_{j=1}^{J} x_j = q \right\}, \tag{1}$$

$$n(q,J) = |C(q,J)| = \binom{J+q-1}{q}, \tag{2}$$

$$\tilde{C}(2q,J) = \{(\bar{x},\bar{y}) | \bar{x},\bar{y} \in C(q,J)\}, \tag{3}$$

$$\tilde{C}_l(2q,J) = \left\{ (\bar{x},\bar{y}) \in \tilde{C}(2q,J) \;\middle|\; \sum_{j=1}^{J} |x_j - y_j| = 2l \right\}, \tag{4}$$

$$\hat{C}(2q,J) = \left\{ (\bar{x},\bar{y}) \in \tilde{C}(2q,J) | x_j y_j = 0 \text{ for } 1 \leqslant j \leqslant J \right\}. \tag{5}$$

Further, let

$$m(l,J) = \frac{1}{2} \sum_{r=1}^{l} \binom{J}{r} \binom{l-1}{r-1} \binom{J-r+l-1}{l} \tag{6}$$

and

$$M(l,J) = \sum_{r=1}^{l} m(r,J). \tag{7}$$

We define the function μ on $\tilde{C}(2q,J)$ by

$$\mu(\bar{x},\bar{y}) = (\bar{u},\bar{v}), \tag{8}$$

where

$$u_j = x_j - \min(x_j, y_j), \qquad v_j = y_j - \min(x_j, y_j), \tag{9}$$

and we define an equivalence on $\tilde{C}_l(2q,J)$ by

$$(\bar{x},\bar{y}) \equiv (\bar{x}',\bar{y}') \text{ if and only if } \mu(\bar{x},\bar{y}) = \mu(\bar{x}',\bar{y}'). \tag{10}$$

From [4] we quote some results without proof.

Lemma 1. (a) *If* $(\bar{x},\bar{y}) \in \tilde{C}_l(2q,J)$, *then* $\mu(\bar{x},\bar{y}) \in \hat{C}(2l,J)$,

(b) *Each* $(\bar{u},\bar{v}) \in \hat{C}(2l,J)$ *is the image under* μ *of exactly* $n(q-l,J)$ *(equivalent) elements in* $\tilde{C}_l(2q,J)$, *namely* $\{(\bar{u}+\bar{z},\bar{v}+\bar{z}) | \bar{z} \in C(q-l,J)\}$.

(c) *For* $l > 0$ *we have* $|\hat{C}(2l,J)| = 2m(l,J)$.

Let $A = \{a_1,a_2,\ldots,a_J\}$ be a $(2q,J)$-DS, where $a_1 < a_2 < \cdots < a_J$ and let

$$\hat{A} = \{b_0,b_1,\ldots,b_{\hat{J}}\} \stackrel{\text{def}}{=} \left\{ \sum_{j=1}^{J} x_j a_j \middle| (x_1,x_2,\ldots,x_J) \in C(q,J) \right\}, \tag{11}$$

where $\hat{J} = n(q,J) - 1$ and $b_0 < b_1 < \ldots < b_{\hat{J}}$. Then

$$b_0 = qa_1, \quad b_{\hat{J}} = qa_J. \tag{12}$$

Consider the differences between the elements in $\hat{A}$. Define

$$\sigma_A(\bar{x},\bar{y}) \stackrel{\text{def}}{=} \sum_{j=1}^{J} x_j a_j - \sum_{j=1}^{J} y_j a_j = \sum_{j=1}^{J} (x_j - y_j) a_j, \tag{13}$$

and

$$\tilde{C}^{+}(2q,J) = \left\{ (\bar{x},\bar{y}) \in \tilde{C}(2q,J) \;\middle|\; \sigma_A(\bar{x},\bar{y}) > 0 \right\}, \tag{14}$$

and $\tilde{C}_l^{+}(2q,J)$ similarly. Then the multiset of positive differences are given by

$$\left\{ \sigma_A(\bar{x},\bar{y}) | (\bar{x},\bar{y}) \in \tilde{C}^{+}(2q,J) \right\}. \tag{15}$$

In particular, if $(\bar{x},\bar{y}) \equiv (\bar{x}',\bar{y}')$, then by Lemma 1(b), $\sigma_A(\bar{x},\bar{y}) = \sigma_A(\bar{x}',\bar{y}')$, independent of A. We define a (q,ρ,I,J)*-multiple difference set of a distinct sum set* (MDSDS) to be a set

$$\mathscr{A} = \{A_1,A_2,\ldots,A_I\} \tag{16}$$

where $A_1,A_2,\ldots,A_I$ are $(2q,J)$-DS such that

$$\sum_{i=1}^{I} \sum_{l=1}^{q} \frac{\left| \left\{ (\bar{x},\bar{y}) \in \tilde{C}_l^{+}(2q,J) \mid \sigma_{A_i}(\bar{x},\bar{y}) = k \right\} \right|}{n(q-l,J)} \leqslant \rho \tag{17}$$

for all k.

Let

$$m(\mathscr{A}) = \max\{a_{iJ} | 1 \leqslant i \leqslant I, 1 \leqslant j \leqslant J\}. \tag{18}$$

Finally, let

$$M(q,\rho,I,J) = \min\{m(\mathscr{A}) | \mathscr{A} \text{ is a } (q,\rho,I,J) - \text{MDSDS}\}. \tag{19}$$

Our goal is to find lower bounds on $M(q,\rho,I,J)$. A general result is given in Theorem 1. Specializations and asymptotic results are given in subsequents theorems.

An important special case is $M(q,1,1,J) = N_{2q}(1,J)$ where $N_{2q}(1,J)$ is the smallest maximal element of a DS. This has been studied extensively for $q = 1$ and asymptotically tight bounds are known. For $q > 1$ much less is known. Theorems 2 and 4 give better lower bounds on $N_{2q}(1,J)$ for $q > 2$ than the previous best lower bounds which were given in [4,6].

Putting $I = 1$, $\rho \geqslant 1$, $q \geqslant 1$, and $h = 2q$ in Theorems 1-4 we get a generalization of the lower bounds on multiple DS given in [4,6].

3. A general lower bound

Our proof of a lower bound is a modification of the proofs in [5,9]. Let $\mathscr{A}$ be an (q,ρ,I,J)-MDSDS. Let

$$S_k = \sum_{i=1}^{I} \sum_{l=0}^{\hat{J}-k} (b_{i,k+l} - b_{il}), \tag{20}$$

$$\bar{S}_k = \sum_{i=1}^{I} \sum_{l=0}^{k-1} \left(m - (b_{i,\hat{J}+1-k+l} - b_{il})\right), \tag{21}$$

$$d_{ir} = b_{i,r+1} - b_{ir}, \tag{22}$$

$$g_r = \sum_{i=1}^{I} \left(d_{ir} + d_{i,\hat{J}-1-r}\right). \tag{23}$$

Let t be a positive integer, and let $\bar{\alpha} = (\alpha_2, \alpha_3, \ldots, \alpha_t)$ be a sequence of real numbers. Define

$$\alpha_1 = C - \sum_{k=2}^{t} k\alpha_k, \tag{24}$$

where

$$C = \binom{t+1}{2}. \tag{25}$$

For $0 \leqslant r \leqslant t-2$, define

$$\beta_r = \binom{t-r}{2} - \sum_{k=r+2}^{t} (k-1-r)\alpha_k. \tag{26}$$

From [5, Lemma 4] we quote the following result without proof.

Lemma 2. *Let* $t \leqslant \hat{J}/2 + 1$. *Then*

$$CIm = \sum_{k=1}^{t} \alpha_k \bar{S}_k + \sum_{k=1}^{t} S_k + \sum_{r=0}^{t-2} \beta_r g_r.$$

The idea is to lower bound each term in Lemma 2. This we can do if $\bar{\alpha}$ is *feasible*, i.e.

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_t \geqslant 0, \tag{27}$$

and

$$\beta_0 \geqslant \beta_1 \geqslant \cdots \geqslant \beta_{t-2} \geqslant 0. \tag{28}$$

These conditions can be simplified. Some simplification was done in [5, Lemma 6], however, it can be simplified further.

Lemma 3. *The sequence* $(\alpha_2, \alpha_3, \ldots, \alpha_t)$ *is feasible if and only if*

$$\alpha_2 \geqslant \alpha_3 \geqslant \cdots \geqslant \alpha_t \geqslant 0, \tag{29}$$

and

$$\sum_{k=2}^{t} \alpha_k \leqslant t - 1. \tag{30}$$

Proof. First we prove the 'only if' part. Clearly (27) implies (29). We observe that

$$\beta_r - \beta_{r+1} = t - r - 1 - \sum_{k=r+2}^{t} \alpha_k. \tag{31}$$

In particular, $\beta_0 - \beta_1 = t - 1 - \sum_{k=2}^{t} \alpha_k$, and so (28) implies (30).

Next we prove the 'if' part. Let $0 \leqslant r \leqslant t-2$. Then (29) and (30) implies that

$$\begin{aligned}
(t-r-1)(t-1) &\geqslant (t-r-1)\sum_{k=2}^{t} \alpha_k \\
&= (t-r-1)\sum_{k=2}^{r+1} \alpha_k + (t-1)\sum_{k=r+2}^{t} \alpha_k - r\sum_{k=r+2}^{t} \alpha_k \\
&\geqslant (t-r-1) r \alpha_{r+1} + (t-1)\sum_{k=r+2}^{t} \alpha_k - r(t-r-1)\alpha_{r+2} \\
&\geqslant (t-1)\sum_{k=r+2}^{t} \alpha_k.
\end{aligned}$$

Hence, we get

$$\sum_{k=r+2}^{t} \alpha_k \leqslant t - r - 1. \tag{32}$$

By (31), this implies (28). It remain to show that $\alpha_1 \geqslant \alpha_2$. However, by (32) we get

$$\begin{aligned}\alpha_1 - \alpha_2 &= C - \left\{\alpha_2 + \sum_{k=2}^{t} k\alpha_k\right\}\\ &= C - \left\{3\sum_{k=2}^{t}\alpha_k + \sum_{r=2}^{t}\sum_{k=r+2}^{t}\alpha_k\right\}\\ &\geqslant C - 3(t-1) - \sum_{r=2}^{t}(t-r-1)\\ &= C - 3(t-1) - \frac{(t-3)(t-2)}{2} = C - \frac{t(t+1)}{2} = 0. \qquad \square\end{aligned}$$

For any integer M, let

$$\langle M\rangle = \langle M\rangle_\rho = \left(M - \frac{\rho}{2}\left\lfloor\frac{M}{\rho}\right\rfloor\right)\left(\left\lfloor\frac{M}{\rho}\right\rfloor + 1\right). \tag{33}$$

In [5] we showed the following results:

Lemma 4. (a) *For any M we have*

$$M(M+\rho)/2\rho \leqslant \langle M\rangle \leqslant M(M+\rho)/2\rho + \rho/8,$$

(b) *if $(u_1, u_2, \ldots, u_m)$ is a sequence of positive integers such that any integer appears at most ρ times in the sequence, then*

$$\sum_{j=1}^{M} u_j \geqslant \langle M\rangle,$$

(c) *if $(u_1, u_2, \ldots, u_m)$ is a sequence of non-negative integers such that any integer appears at most ρ times in the sequence, then*

$$\sum_{j=1}^{M} u_j \geqslant \langle M - \rho\rangle = \langle M\rangle - M.$$

From now on we use the following notations. For a given V, let s be defined by

$$\sum_{i=1}^{s} n(q-i,J)m(i,J)I < V \leqslant \sum_{i=1}^{s+1} n(q-i,J)m(i,J)I. \tag{34}$$

Let

$$\begin{aligned}\{V\} &= \sum_{i=1}^{s} n(q-i,J-1)\langle M(i,J)I - \rho\rangle_\rho\\ &\quad + \left\langle V - \sum_{i=1}^{s} n(q-i,J)m(i,J)I + n(q-s-1,J)M(s,J)I - \rho_s\right\rangle_{\rho_s}, \qquad (35)\\ \{V\}' &= \{V\} + V, \qquad (36)\end{aligned}$$

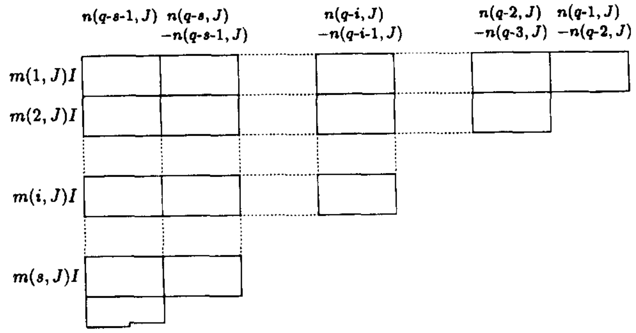

Fig. 1. Splitting in subsets.

where

$$\rho_s = n(q-s-1,J)\rho. \tag{37}$$

Lemma 5. (a) *If $\mathscr{V}$ is a submultiset of the multiset*

$$\{b_{ij} - b_{ij'} | 0 \leqslant j' < j \leqslant \hat{J}, 1 \leqslant i \leqslant I\} \tag{38}$$

with V non-negative elements, then

$$\sum_{v \in \mathscr{V}} v \geqslant \{V\}.$$

(b) *If $\mathscr{V}$ is a submultiset of the multiset* (38) *with V positive elements, then*

$$\sum_{v \in \mathscr{V}} v \geqslant \{V\}'.$$

Proof. First, consider the submultisets of (38) with $n(q-1,J)$ equal elements. By Lemma 1(c) there are at most $m(1,J)I$ such sets. Their sums is smallest if their elements are $\{0,1,\ldots,m(1,j)I-1\}$. Next, there are at most $m(2,J)I$ submultisets with $n(q-2,J)$ equal elements, etc. This is illustrated in Fig 1.

We get a lower bound for $\sum_{v \in \mathscr{V}} v$ as the sum of the elements in the array with all elements in each row equal and the elements in different rows different. The sum of the elements in the $n(q-i,J) - n(q-i-1,J)$ columns indicated in the figure is lower bounded by

$$(n(q-i,J) - n(q-i-1,J))\langle M(i,J)I - \rho\rangle_\rho$$

for $1 \leqslant i \leqslant s$. Note that $n(q-i,J) - n(q-i-1,J) = n(q-i,J-1)$. Finally, in the leftmost $n(q-s-1,J)$ columns there are

$$V - \sum_{i=1}^{s} n(q-i,J)m(i,J)I + n(q-s-1,J)M(s,J)I$$

elements, each appearing at most $n(q-s-1,J)\rho = \rho_s$ times, and so their sum is lower bounded by

$$\left\langle V - \sum_{i=1}^{s} n(q-i,J)m(i,J)I + n(q-s-1,J)M(s,J)I - \rho_s \right\rangle_{\rho_s}.$$

Summing the lower bounds, the lemma follows. □

Lemma 6. *If $\bar{\alpha}$ is feasible, then*

$$\sum_{k=1}^{t} \alpha_k \bar{S_k} \geqslant \sum_{k=1}^{t} \alpha_k(\{V_k\} - \{V_{k-1}\}),$$

where

$$V_k = k(k+1)I/2. \tag{39}$$

Proof. Let

$$\bar{T}_k = \sum_{r=1}^{k} \bar{S}_r = \sum_{r=1}^{k} \sum_{i=1}^{I} \sum_{l=0}^{r-1} (m - (b_{i,\hat{J}+1-r+l} - b_{il})). \tag{40}$$

$\bar{T}_k$ is a sum of V_k non-negative integers. As in the proof of Lemma 5 we get

$$\bar{T}_k \geqslant \{V_k\}.$$

Let $\alpha_{t+1} = 0$. We have

$$\begin{aligned} \sum_{k=1}^{t} \alpha_k \bar{S_k} &= \sum_{k=1}^{t} \alpha_k(\bar{T}_k - \bar{T}_{k-1}) \\ &= \sum_{k=1}^{t} (\alpha_k - \alpha_{k+1})\bar{T}_k \\ &\geqslant \sum_{k=1}^{t} (\alpha_k - \alpha_{k+1})\{V_k\} \\ &= \sum_{k=1}^{t} \alpha_k(\{V_k\} - \{V_{k-1}\}). \qquad \square \end{aligned}$$

Lemma 7. *Let*

$$U_t = I(t\hat{J} - t(t-1)/2).$$

For all $t \geqslant 1$ we have $\sum_{k=1}^{t} S_k \geqslant \{U_t\}'$.

Proof. $\sum_{k=1}^{t} S_k$ is a sum of positive integers. The number of terms in the sum is

$$I\hat{J} + I(\hat{J}-1) + \cdots + I(\hat{J}-t+1) = I(t\hat{J} - t(t-1)/2) = U_t.$$

Hence, the lemma follows from Lemma 5. □

Lemma 8. *Let $t \leqslant \hat{J}/2+1$ and let $\bar{\alpha}$ be feasible. Then*

$$\sum_{r=0}^{t-2} \beta_r g_r \geqslant \sum_{r=1}^{t-1} (t-r)\{2rI\}' - \sum_{k=2}^{t} \alpha_k \sum_{r=1}^{k-1} \{2rI\}'.$$

Proof. Let $\beta_{t-1}=0$ and let

$$h_r = \sum_{l=0}^{r-1} g_l = \sum_{l=0}^{r-1} \sum_{i=1}^{I} (d_{il} + d_{i,J-1-l});$$

h_r is a sum of $2rI$ positive integers. By Lemma 5 we get

$$h_r \geqslant \{2rI\}'. \tag{41}$$

By (31) and (41) we get

$$\begin{aligned}
\sum_{r=0}^{t-2} \beta_r g_r &= \sum_{r=0}^{t-2} \beta_r (h_{r+1} - h_r) \\
&= \sum_{r=1}^{t-1} (\beta_{r-1} - \beta_r) h_r \\
&\geqslant \sum_{r=1}^{t-1} \Big(t - r - \sum_{k=r+1}^{t} \alpha_k\Big)\{2rI\}' \\
&= \sum_{r=1}^{t-1} (t-r)\{2rI\}' - \sum_{k=2}^{t} \alpha_k \sum_{r=1}^{k-1} \{2rI\}'. \qquad \square
\end{aligned}$$

Combining Lemmas 2, 6–8 we get the following theorem.

Theorem 1. *Let*

$$f_k = \{V_k\} - \{V_{k-1}\} - k\{V_1\} - \sum_{r=1}^{k-1} \{2rI\}'.$$

If $t \leqslant \hat{J}/2+1$ and $\bar{\alpha}$ is feasible, then

$$M(q,\rho,I,J) \geqslant \frac{1}{qIC}\left(\{U_t\}' + \sum_{r=1}^{t-1} (t-r)\{2rI\}' + C\{V_1\} + F(\bar{\alpha}) \right),$$

where

$$F(\bar{\alpha}) = \sum_{k=2}^{t} f_k \alpha_k.$$

For a given t, the best lower bound is obtained for a feasible $\bar{\alpha}$ which maximizes $F(\bar{\alpha})$. Such an $\bar{\alpha}$ is called *optimal* (it may not be unique). We will now find an optimal $\bar{\alpha}$.

Lemma 9. *Let $t \leqslant \hat{J}/2+1$. There exists a K, $2 \leqslant K \leqslant t$ such that $\bar{\alpha}^*$ defined by*

$$\alpha_k^* = \frac{t-1}{K-1} \quad \textit{for } 2 \leqslant k \leqslant K,$$

$$\alpha_k^* = 0 \quad \textit{for } K < k \leqslant t,$$

is optimal.

Proof. Let $\bar{\alpha} = (\alpha_2, \alpha_3, \ldots, \alpha_t)$ be optimal, and let K be defined by $\alpha_K > 0$ and $\alpha_k = 0$ for $k > K$. If there is more than one optimal $\bar{\alpha}$, choose one with smallest value of the corresponding K. If $F(\bar{\alpha}) = 0$, $\bar{\alpha} = (0, 0, \ldots, 0)$ and we are finished. Therefore, assume that $F(\bar{\alpha}) > 0$. Suppose $\alpha_i > \alpha_{i+1}$ for some $i < K$. Define $\bar{\alpha}'$ by

$$\alpha_k' = \alpha_k - (K-i)\varepsilon \quad \text{for } 2 \leqslant k \leqslant i,$$

$$\alpha_k' = \alpha_k + (i-1)\varepsilon \quad \text{for } i+1 \leqslant k \leqslant K,$$

$$\alpha_k' = 0 \quad \text{for } k > K.$$

where

$$-\frac{\alpha_K}{i-1} \leqslant \varepsilon \leqslant \frac{\alpha_i - \alpha_{i+1}}{K-1}.$$

Then $\bar{\alpha}'$ is feasible and

$$F(\bar{\alpha}') = F(\bar{\alpha}) + \varepsilon \left((i-1) \sum_{k=i+1}^{K} f_k - (K-i) \sum_{k=2}^{i} f_k \right).$$

Hence $(i-1)\sum_{k=i+1}^{K} f_k - (K-i)\sum_{k=2}^{i} f_k = 0$, since otherwise we can increase the sum by a suitable choice of ε. In particular, $\bar{\alpha}'$ is also optimal. However, if we choose $\varepsilon = -\alpha_K/(i-1)$ we get $\alpha_k' = 0$ for $k \geqslant K$, contradicting the minimality of K. Therefore, $\alpha_i = \alpha_{i+1}$ for $2 \leqslant i < K$. Let

$$\sum_{k=2}^{t} \alpha_k = S.$$

If $S < t-1$, then $(t-1)/S\bar{\alpha}$ is also feasible and

$$F\left(\frac{t-1}{S}\bar{\alpha}\right) = \frac{t-1}{S} F(\bar{\alpha}) > F(\bar{\alpha}),$$

a contradiction. Hence $S = t-1$ and so $\alpha_k = (t-1)/(K-1)$ for $2 \leqslant k \leqslant K$. $\square$

From Lemma 9 we immediately get the next lemma.

Lemma 10. *If $t \leqslant \hat{J}/2+1$ and $\bar{\alpha}^*$ is optimal, then*

$$F(\bar{\alpha}^*) = \max_{2\leqslant k_0\leqslant t}\left(\frac{t-1}{k_0-1}\sum_{k=2}^{k_0} f_k, 0\right)$$
$$= \max_{2\leqslant k_0\leqslant t}\left(\frac{t-1}{k_0-1}\left(\{V_{k_0}\} - \frac{k_0(k_0+1)}{2}\{V_1\} - \sum_{k=2}^{k_0}\sum_{r=1}^{k-1}\{2rI\}'\right), 0\right).$$

Lemma 11. *Let $q>1$, $t^* = \lfloor J^{q-0.5}/(q-1)!\rfloor + l$, $l=-1,0,1$, and J be large enough. Then*

(a) $\sum_{k=2}^{t^*} f_k = \frac{1}{8\rho(q-1)!^4} I^2 J^{4q-3} - \frac{1}{6\rho(q-1)!^3} I^2 J^{3q-1} + O(I^2J^{4q-4}) + O(I^2J^{3q-1.5})$,

(b) $\frac{t-1}{k_0-1}\sum_{k=2}^{k_0} f_k < 0$

if $q=2$, $k_0 \leqslant t \leqslant t^*$, and $V_{k_0} > IJ^2(J-1)/2$,

(c) $F(\bar{\alpha}^*) = \sum_{k=2}^{t} f_k$

if $q>2$, $J=O(k_0)$, $t<\hat{J}/2$, $V_t > (1-\varepsilon)\sum_{i=0}^{q-1} n(q-i,J)m(i,J)I$, and $1>\varepsilon>0$.

Proof. (a) From the definitions of {M} and {M}' we have

$$\{M\} = a_s + \langle M - b_s + c_s - \rho_s\rangle_{\rho_s}, \tag{42}$$

$$\{M\}' = \{M\} + M, \tag{43}$$

where

$$a_s = \sum_{i=0}^{s} n(q-i,J-1)\langle M(i,J)I - \rho\rangle = \frac{1}{8\rho s!^4(q-s)!} I^2 J^{3s+q}\left(1+O(J^{-1})\right),$$

$$b_s = \sum_{i=0}^{s} n(q-i,J)m(i,J)I = \frac{1}{2s!^2(q-s)!} IJ^{q+s}\left(1+O(J^{-1})\right),$$

$$c_s = n(q-s-1,J)M(s,J)I = \frac{1}{2s!^2(q-s-1)!} IJ^{q+s-1}\left(1+O(J^{-1})\right),$$

$$\rho_s = n(q-s-1,J)\rho = \frac{\rho}{(q-s-1)!} J^{q-s-1}\left(1+O(J^{-1})\right),$$

$$b_s < M \leqslant b_{s+1}.$$

By (42) and Lemma 4(a) we get

$$\{V_k\} = a_s + \frac{1}{2\rho_s}(V_k - b_s + c_s)^2 - \frac{1}{2}(V_k - b_s + c_s) + e_1(k), \tag{44}$$

where

$$b_s < V_k \leqslant b_{s+1}, \qquad 0 \leqslant e_1(k) \leqslant \rho_s/8.$$

Further, note that

$$r < t < \hat{J}/2, \qquad 2rI < b_1$$

and so the corresponding s is 0. By (43) and Lemma 4(a) we have

$$\sum_{r=1}^{k-1}\{2rI\}' = \sum_{r=1}^{k-1}\langle 2rI\rangle_{\rho_0} = \sum_{r=1}^{k-1}\left(\frac{2rI(2rI+\rho_0)}{2\rho_0} + e_2(r)\right),$$

$$\sum_{k=2}^{k_0}\sum_{r=1}^{k-1}\{2rI\}' = \frac{1}{6\rho_0}I^2k_0^2(k_0^2-1) + \frac{1}{6}Ik_0(k_0^2-1) + \sum_{k=2}^{k_0}\sum_{r=1}^{k-1}e_2(r), \tag{45}$$

where $0 \leqslant e_2(r) \leqslant \rho_0/8$. By (42) and Lemma 4(a) we have

$$\frac{1}{2}k_0(k_0+1)\{V_1\} = \frac{1}{2}k_0(k_0+1)\langle I-\rho_0\rangle_{\rho_0} = \frac{1}{2}k_0(k_0+1)\left(\frac{I^2}{2\rho_0} - \frac{I}{2} + e_3\right), \tag{46}$$

where $0 \leqslant e_3 \leqslant \rho_0/8$.

Hence, by (44)–(46) and Lemma 10 we have

$$\frac{t-1}{k_0-1}\sum_{k=2}^{k_0} f_k = g_1(k_0) + h_1(k_0), \tag{47}$$

where

$$g_1(k_0) = \frac{t-1}{k_0-1}\left(a_s + \frac{1}{2\rho_s}(V_{k_0} - b_s + c_s)^2 - \frac{1}{2}(V_{k_0} - b_s + c_s)\right.$$
$$\left. - \frac{1}{6\rho_0}I^2k_0^2(k_0^2-1) - \frac{1}{6}Ik_0(k_0^2-1) - \frac{1}{2}k_0(k_0+1)\left(\frac{I^2}{2\rho_0} - \frac{I}{2}\right)\right), \tag{48}$$

$$h_1(k_0) = \frac{t-1}{k_0-1}\left(e_1(k_0) - \sum_{k=2}^{k_0}\sum_{r=1}^{k-1}e_2(r) - \frac{1}{2}k_0(k_0+1)e_3\right). \tag{49}$$

Note that

$$V_{t^*} = \frac{1}{2(q-1)!^2}IJ^{2q-1}(1+O(J^{-1})) \quad\text{and}\quad V_{t^*} - b_{q-1} = O(IJ^{2q-2}).$$

Let $k_0 = t^*$. Substituting the value for t^* in (47)-(49) we get

$$\sum_{k=2}^{t^*} f_k = \frac{1}{8\rho(q-1)!^4}I^2J^{4q-3} - \frac{1}{6\rho(q-1)!^3}I^2J^{3q-1} + O(I^2J^{4q-4}) + O(I^2J^{3q-1.5}).$$

(b) If $q=2$, $k_0 \leqslant t \leqslant t^*$, and $V_{k_0} > IJ^2(J-1)/2$, then

$$V_{k_0} = IJ^3(1+O(J^{-1}))/2 \quad\text{and}\quad k_0 = J^{1.5}(1+O(J^{-1})).$$

By (47)-(49) we get

$$\frac{t-1}{k_0-1}\sum_{k=2}^{k_0} f_k = -\frac{1}{24\rho}I^2J^5 + O(I^2J^{4.5}) < 0.$$

(c) Let $s > 0$ and k_0 be real numbers, $J = \mathrm{O}(k_0)$. Then $k_0 - 1 = k_0(1 + \mathrm{O}(J^{-1}))$. Differentiating (48) and simplifying, we get

$$\frac{d(g_1(k_0))}{dk_0} = (t-1)\left(\frac{1}{k_0^2}\left(\frac{I}{\rho_s}(V_{k_0} - b_s + c_s)\left(k_0^2 - \frac{1}{2I}(V_{k_0} - b_s + c_s)\right) - a_s\right)\right.$$
$$\left. - \left(\frac{1}{2\rho_0}I^2k_0^2 + \frac{1}{3}Ik_0\right)\right)(1 + \mathrm{O}(J^{-1})). \tag{50}$$

Let $V_{k_0} = b_s + d$, $d > 0$. We bound each term in $d(g_1(k_0))/dk_0$ and get

$$\frac{I}{k_0^2\rho_s}(V_{k_0} - b_s + c_s)\left(k_0^2 - \frac{1}{2I}(V_{k_0} - b_s + c_s)\right) \geqslant \begin{cases} L_1 & \text{if } d = \mathrm{O}(c_s), \\ L_2 & \text{if } c_s = \mathrm{o}(d), \end{cases}$$

where

$$L_1 = c_sI(1 + \mathrm{O}(J^{-1}))/\rho_s,$$

$$L_2 = 3dI(1 + \mathrm{O}(J^{-1}))/4\rho_s,$$

$$\frac{a_s}{k_0^2} < \frac{I^2J^{2s}}{8\rho s!^2}(1 + \mathrm{O}(J^{-1})) < \frac{I^2J^{2s}}{2\rho s!^2}(1 + \mathrm{O}(J^{-1})) = L_1,$$

$$\frac{1}{2\rho_0}I^2k_0^2 = \begin{cases} b_sI(1 + \mathrm{O}(J^{-1}))/\rho_0 \ll 3L_1/4 & \text{if } d = \mathrm{o}(b_s),\ s > 1, \\ (c+1)b_sI(1 + \mathrm{O}(J^{-1}))/\rho_0 \ll L_2 & \text{if } d \sim cb_s, c > 0, \\ dI(1 + \mathrm{O}(J^{-1}))/\rho_0 \ll L_2 & \text{if } b_s = \mathrm{o}(d), \end{cases}$$

$$\frac{1}{3}Ik_0 = \begin{cases} (2b_sI)^{0.5}(1 + \mathrm{O}(J^{-1}))/3 \ll 3L_1/4 & \text{if } d = \mathrm{o}(b_s),\ s > q/3, \\ (2(c+1)b_sI)^{0.5}(1 + \mathrm{O}(J^{-1}))/3 \ll L_2 & \text{if } d \sim cb_s,\ s > (q-2)/3, \\ (2dI)^{0.5}(1 + \mathrm{O}(J^{-1}))/3 \ll L_2 & \text{if } b_s = \mathrm{o}(d), s > (q-2)/3, \end{cases}$$

where $c > 0$. Hence

$$\frac{d(g_1(k_0))}{dk_0} \geqslant \frac{3tI^2J^{2s}}{8\rho s!^2}(1 + \mathrm{O}(J^{-1})) > 0 \tag{51}$$

if $d = \mathrm{o}(b_s)$, $s > q/3$ or $d \geqslant cb_s$, $c > 0$, $s > (q-2)/3$.

By (49) and

$$\frac{1}{k_0}\sum_{k=2}^{k_0+1}\sum_{r=1}^{k-1} e_2(r) - \frac{1}{k_0 - 1}\sum_{k=2}^{k_0}\sum_{r=1}^{k-1} e_2(r) = \frac{1}{k_0}\sum_{r=1}^{k_0} e_2(r) - \frac{1}{k_0(k_0-1)}\sum_{k=2}^{k_0}\sum_{r=1}^{k-1} e_2(r),$$

we get

$$h_1(k_0 + 1) - h_1(k_0) = \mathrm{O}(tJ^{q-1}). \tag{52}$$

From (47), (51), and (52) we have

$$\frac{t-1}{k_0}\sum_{k=2}^{k_0+1} f_k - \frac{t-1}{k_0 - 1}\sum_{k=2}^{k_0} f_k > 0 \tag{53}$$

if $d = \mathrm{o}(b_s), s > (q-1)/2$ or $d \geqslant cb_s, c > 0, s > (q-2)/2$.

Let $q > 2, s = q - 1$ or $s = q - 2, k_0 \leqslant t < \hat{J}/2, V_{k_0} \geqslant (1-\varepsilon)b_{q-1}, 1 > \varepsilon > 0$. By (53) we get

$$\max_{k_0 \leqslant t < \hat{J}/2,\ V_{k_0} \geqslant (1-\varepsilon)b_{q-1}} \frac{t-1}{k_0-1} \sum_{k=2}^{k_0} f_k = \sum_{k=2}^{t} f_k. \tag{54}$$

On the other hand, by (47)–(49) we have

$$\frac{1}{k_0-1}\sum_{k=2}^{k_0} f_k = \frac{1}{k_0}\left(a_s + \frac{1}{2\rho_s}(V_{k_0} - b_s + c_s)^2\right) - \left(\frac{1}{6\rho}I^2k_0^3 + \frac{1}{6}Ik_0^2 + I^2k_0 \mathrm{O}(\rho_0)\right)(1 + \mathrm{O}(J^{-1})).$$

Let $q > 2$ and $V_{k_0} < b_{q-1}$. Since

$$\frac{1}{6\rho}I^2k_0^3 + \frac{1}{6}Ik_0^2 + I^2k_0\mathrm{O}(\rho_0) = \mathrm{O}(I^2J^{2q-0.5}),$$

we have

$$\frac{1}{k_0-1}\sum_{k=2}^{k_0} f_k = \begin{cases} o(I^2J^{2q+s-0.5}) & \text{if } V_{k_0} = \mathrm{o}(b_{q-1}), \\ \dfrac{(1-\varepsilon)^{1.5}}{8\rho(q-1)!^3}I^2J^{3q-2.5}(1+\mathrm{O}(J^{-1})) & \text{if } V_{k_0} \sim (1-\varepsilon)b_{q-1}. \end{cases} \tag{55}$$

If $s = 0$, then

$$\frac{1}{k_0-1}\sum_{k=2}^{k_0} f_k = \mathrm{O}(I^2J^{2q}). \tag{56}$$

By (54)–(56) we get

$$F(\bar{\alpha}^*) = \sum_{k=2}^{t} f_k$$

if $q > 2,\ t < \hat{J}/2,\ V_t > (1-\varepsilon)b_{q-1},\ 1 > \varepsilon > 0$. □

From Theorem 1 and Lemma 11(c) we get the following theorem.

Theorem 2. *Let $q > 2,\ t < \hat{J}/2,\ V_t > (1-\varepsilon)b_{q-1},\ 1 > \varepsilon > 0$, and let J be large enough. Then the optimal $\bar{\alpha}$ in Theorem 1 is $\alpha_k = 1$ for $1 \leqslant k \leqslant t$, i.e. $\beta_r = 0$ for all r. We have*

$$M(q,\rho,I,J) \geqslant \frac{1}{qIC}(\{U_t\}' + \{V_t\}).$$

Lemma 12. *If $q = 2$ and $V_k \leqslant IJ^2(J-1)/2$, then $f_k < 0$ for $k \geqslant 5$.*

Lemma 13. *Let $q = 2$. We have*

(a) $f_2 \geqslant 0 \Leftrightarrow I \geqslant \rho J$,

(b) $f_3 \geqslant f_2 \Leftrightarrow I \geqslant 2\rho J$,

(c) $f_3 \geqslant f_4$, *always*,

(d) $2f_4 - f_3 - f_2 \geqslant 0 \Leftrightarrow I \geqslant 8\rho J$.

Changing ρ into ρJ in the proofs of Lemmas 13 and 14 in [5] we get proofs of Lemmas 12 and 13.

Lemma 14. *Let $q = 2$, $t \leqslant t^*$, and let J be large enough. Then the following conditions give optimal $\bar{\alpha}$ in the various cases:*

(a) *if* $I \leqslant \rho J$:	$\alpha_k = 0$ *for* $2 \leqslant k \leqslant t$,
(b) *if* $t = 2$ *and* $I > \rho J$:	$\alpha_2 = 1$,
(c) *if* $t = 3$ *and* $\rho J < I < 2\rho J$:	$\alpha_2 = 2$, $\alpha_3 = 0$,
if $t = 3$ *and* $I \geqslant 2\rho J$:	$\alpha_2 = \alpha_3 = 1$,
(d) *if* $t \geqslant 4$ *and* $\rho J < I < 2\rho J$:	$\alpha_2 = t - 1$,
	$\alpha_k = 0$ *for* $3 \leqslant k \leqslant t$,
(e) *if* $t \geqslant 4$ *and* $2\rho J \leqslant I < 8\rho J$:	$\alpha_2 = \alpha_3 = (t-1)/2$,
	$\alpha_k = 0$ *for* $4 \leqslant k \leqslant t$,
(f) *if* $t \geqslant 4$ *and* $8\rho J \leqslant I$:	$\alpha_2 = \alpha_3 = \alpha_4 = (t-1)/3$,
	$\alpha_k = 0$ *for* $5 \leqslant k \leqslant t$.

Proof. (e) and (f): The other cases are similar. If $k_0 \leqslant t^*$ and $V_{k_0} \geqslant IJ^2(J-1)/2$, then by Lemma 11(b), we have

$$\max_{k' \leqslant k_0 \leqslant t} \frac{t-1}{k_0 - 1} \sum_{k=2}^{k_0} f_k < 0,$$

where

$$V_{k'} > IJ^2(J-1)/2 \geqslant V_{k'-1}.$$

Hence, by Lemma 10 we get

$$F(\bar{\alpha}^*) = \max_{2 \leqslant k_0 \leqslant k'} \left(\frac{t-1}{k_0 - 1} \sum_{k=2}^{k_0} f_k, 0 \right). \tag{57}$$

If $t \geqslant 4$ and $2\rho J \leqslant I < 8\rho J$, then by Lemmas 12 and 13 we have

$$f_3 \geqslant f_2 > 0 > f_k \text{ for } 5 \leqslant k < k',$$

$$f_4 < (f_2 + f_3)/2.$$

Hence

$$(f_2 + f_3 + f_4)/3 < (f_2 + f_3)/2,$$

$$(f_2 + f_3)/2 \geqslant f_2 > 0,$$

and so

$$(f_2+f_3)/2 > \begin{cases} \sum_{k=2}^{4} f_k/3 > \sum_{k=2}^{k_0} f_k/(k_0-1),\ 5\leqslant k<k' & \text{if } \sum_{k=2}^{4} f_k/3>0,\\ 0 > \sum_{k=2}^{k_0} f_k/(k_0-1),\ 5\leqslant k<k' & \text{if } \sum_{k=2}^{4} f_k/3\leqslant 0. \end{cases}$$

By (57) we get (e). If $t\geqslant 4$ and $8\rho J\leqslant I$, then by Lemmas 12 and 13 we have

$$f_3\geqslant f_2>0, \qquad f_4\geqslant (f_2+f_3)/2>0,$$

$$(f_2+f_3+f_4)/3\geqslant \max_{2\leqslant k_0\leqslant k'}\left(\frac{1}{k_0-1}\sum_{k=2}^{k_0} f_k, 0\right).$$

Hence, by (57) we get (f). □

Substituting the optimal values for $\bar{\alpha}$ in $F(\bar{\alpha}^*)$ and combining with Theorem 1 in [5] we get the following theorem.

Theorem 3. *Let $q=1$ or 2, and let J be large enough. Let*

$$\rho_q=\begin{cases}\rho & \text{if } q=1,\\ \rho J & \text{if } q=2.\end{cases}$$

For $t\leqslant t^*$ we have

$$M(q,\rho,I,J)\geqslant \frac{1}{qIC}\left(\{U_t\}'+\sum_{r=1}^{t-1}(t-r)\{2rI\}'+C\{V_1\}+D\right),$$

where

$$D=\begin{cases} 0 \quad \mathit{for}\ t=1\ \mathit{or}\ I\leqslant\rho_q,\\ \langle 3I-\rho_q\rangle_{\rho_q}-3\langle I-\rho_q\rangle_{\rho_q}-\langle 2I\rangle_{\rho_q} \quad \mathit{for}\ t=2\ \mathit{and}\ I>\rho_q,\\ \langle 6I-\rho_q\rangle_{\rho_q}-6\langle I-\rho_q\rangle_{\rho_q}-2\langle 2I\rangle_{\rho_q}-\langle 4I\rangle_{\rho_q} \quad \mathit{for}\ t=3\ \mathit{and}\ I\geqslant 2\rho_q,\\ (t-1)(\langle 3I-\rho_q\rangle_{\rho_q}-3\langle I-\rho_q\rangle_{\rho_q}-\langle 2I\rangle_{\rho_q})\\ \quad \mathit{for}\quad t\geqslant 3\ \mathit{and}\ \rho_q<I<2\rho_q,\\ \dfrac{t-1}{2}(\langle 6I-\rho_q\rangle_{\rho_q}-6\langle I-\rho_q\rangle_{\rho_q}-2\langle 2I\rangle_{\rho_q}-\langle 4I\rangle_{\rho_q})\\ \quad \mathit{for}\quad t\geqslant 4\ \mathit{and}\ 2\rho_q\leqslant I<8\rho_q,\\ \dfrac{t-1}{3}(\langle 10I-\rho_q\rangle_{\rho_q}-10\langle I-\rho_q\rangle_{\rho_q}-3\langle 2I\rangle_{\rho_q}-2\langle 4I\rangle_{\rho_q}-\langle 6I\rangle_{\rho_q})\\ \quad \mathit{for}\quad t\geqslant 4\ \mathit{and}\ I\geqslant 8\rho_q. \end{cases}$$

Theorem 4. *Let $q>1$ and let J be large enough. Then the optimal choice of t in Theorems 1–3 is $t^*=\lfloor J^{q-0.5}/(q-1)!\rfloor+l$, where $l=-1,0$ or 1, and*

$$F(\bar{\alpha}^*)=\frac{1}{8\rho(q-1)!^4}I^2J^{4q-3}-\frac{1}{6\rho(q-1)!^3}I^2J^{3q-1}+O(I^2J^{4q-4})+O(I^2J^{3q-1.5})$$

for $q > 2$, *and* $t = t^*$;

$$M(q,\rho,I,J) \geqslant \frac{I}{q(q!)^2\rho}\Big(J^{2q} - 2qJ^{2q-0.5} + \mathrm{O}(J^{2q-1})\Big).$$

Proof. The proof of Theorem 4 is split into several steps.

(1) From (43) and Lemma 2(e) we have

$$\{U_t\}' = a'_s + \frac{1}{2\rho_s}(U_t - b_s + c_s)^2 - \frac{1}{2}(U_t - b_s + c_s) + e_1(t), \tag{58}$$

where

$$U_t = I(tn(q,J) - t(t+1)/2), b_s < U_t \leqslant b_{s+1}, 0 \leqslant e_1(t) \leqslant \rho_s/8.$$

Let $U_t > b_{q-1}$. Then

$$t > \frac{q}{2(q-1)!}J^{q-1}(1 + \mathrm{O}(J^{-0.5})), \quad s = q-1$$

in (58), and

$$\frac{2}{t(t+1)}\{U_t\}' = \left(\frac{1}{\rho q!^2}I^2J^{2q} - \frac{tI^2J^q}{\rho q!} - \frac{I^2J^{3q-1}}{t\rho q!(q-1)!^2} + \frac{t(t+1)I^2}{4\rho}\right.$$
$$\left. + \frac{I^2J^{4q-2}}{4t(t+1)\rho(q-1)!^4}\right)(1 + \mathrm{O}(J^{-1})) + \frac{2}{t(t+1)}e_1(t). \tag{59}$$

(2) First, let $q > 2$ and $V_t > (1-\varepsilon)b_{q-1}$. By (43) and Lemma 2(e) we have

$$\{V_t\} = a_s + \frac{1}{2\rho_s}(V_t - b_s + c_s)^2 - \frac{1}{2}(V_t - b_s + c_s) + e_2(t),$$

where $s = q-1$ or $q-2$ and $0 \leqslant e_2(t) \leqslant \rho_s/8$. Hence

$$\frac{2}{t(t+1)}\{V_t\} = \left(\frac{t(t+1)I^2}{4\rho_s} + \frac{b_s^2}{t(t+1)\rho_s} - \frac{b_s}{\rho_s} + \frac{e_2(t)}{t(t+1)}\right)(1 + \mathrm{O}(J^{-1})). \tag{60}$$

By (59), (60), and Theorem 2 we have

$$M(q,\rho,I,J) \geqslant \frac{1}{qIC}(\{U_t\}' + \{V_t\})$$
$$= \frac{I}{q(q!)^2\rho}(J^{2q} + \mathrm{O}(J^{2q-1})) + \frac{1}{qI}\Big(g_2(t)(1 + \mathrm{O}(J^{-1})) + h_2(t)\Big), \tag{61}$$

where

$$g_2(t) = -\frac{I^2J^q}{\rho q!}\left(t + \frac{J^{2q-1}}{t(q-1)!^2}\right) + \frac{t(t+1)I^2}{4}\left(\frac{1}{\rho} + \frac{1}{\rho_s}\right)$$
$$+ \frac{1}{t(t+1)}\left(\frac{I^2J^{4q-2}}{4\rho(q-1)!^4} + \frac{b_s^2}{\rho_s}\right), \tag{62}$$

$$h_2(t) = 2(e_1(t) + e_2(t))/t(t+1). \tag{63}$$

Let $a = J^{q-0.5}/(q-1)!$ and t be real numbers. We have

$$\frac{\mathrm{d}(g_2(t))}{\mathrm{d}t} = -\frac{I^2J^q}{\rho q!}\left(1-\frac{a^2}{t^2}\right) + (2t+1)\left(\frac{I^2}{4}\left(\frac{1}{\rho}+\frac{1}{\rho_s}\right)\right.$$
$$\left. -\frac{1}{t^2(t+1)^2}\left(\frac{I^2a^4}{4\rho}+\frac{b_s^2}{\rho_s}\right)\right). \tag{64}$$

Let $t = a+d, d \geqslant 1$. If $t \in [a+1, \hat{J}/2)$, then $s = q-1$ in (60) and we have

$$\frac{\mathrm{d}(g_2(t))}{\mathrm{d}t} = -\frac{I^2J^q}{\rho q!}\left(1-\frac{a^2}{t^2}\right) + \frac{I^2}{2\rho}(2t+1)(1-\frac{a^4}{t^2(t+1)^2})(1+\mathrm{O}(J^{-1}))$$
$$\sim \begin{cases} -2dI^2J^{0.5}/\rho q + 4dI^2/\rho < 0 & \text{if } d = \mathrm{o}(a), \\ -(1-1/c^2)I^2J^q/\rho q! + (2ca+1) & \\ \quad \times(1-1/c^4)I^2/2\rho < 0 & \text{if } t \sim ca, c > 1, \\ -I^2J^q/\rho q! + dI^2/\rho < 0 & \text{if } a = \mathrm{o}(d). \end{cases} \tag{65}$$

Next, let $t = a-d$, $d \geqslant 1$. If $V_t > b_{q-1}$, then $s = q-1$ in (60) and $t \sim a$. Similarly to (65) we have

$$\frac{\mathrm{d}(g_2(t))}{\mathrm{d}t} \sim \frac{2dI^2J^{0.5}}{\rho q} - \frac{4dI^2}{\rho} > 0. \tag{66}$$

If $V_t \leqslant b_{q-1}$, then $s = q-2$ in (60) and by (64) we have

$$\frac{\mathrm{d}(g_2(t))}{\mathrm{d}t} = -\frac{I^2J^q}{\rho q!}(1-\frac{a^2}{t^2}) + \frac{I^2}{4\rho}(2t+1)(1-\frac{a^4}{t^2(t+1)^2})(1+\mathrm{O}(J^{-1}))$$
$$\sim \begin{cases} 2dI^2J^{0.5}/\rho q - 2dI^2/\rho > 0 & \text{if } d = \mathrm{o}(a), \\ -(1-1/c^2)I^2J^q/\rho q! + (2ca+1) & \\ \quad \times(1-1/c^4)I^2/4\rho > 0 & \text{if } t \sim ca,\ c < 1. \end{cases} \tag{67}$$

By (63) we have

$$h_2(t+1) - h_2(t) = \mathrm{O}(\rho_{q-2}/t^2). \tag{68}$$

From (65)–(68) we get

$$(g_2(t+1)-g_2(t))(1+\mathrm{O}(J^{-1})) + h_2(t+1) - h_2(t)$$
$$\begin{cases} < 0 & \text{if } t \in [a+1, \hat{J}/2), \\ > 0 & \text{if } t \leqslant a-1, V_t > cb_{q-1}, \end{cases} \tag{69}$$

where $c = 1-\varepsilon < 1$.

(3) Next, let $q > 2$ and $V_t \leqslant (1-\varepsilon)b_{q-1}$. Then

$$\frac{q}{2(q-1)!}J^{q-1}(1+\mathrm{O}(J^{-0.5})) < t \leqslant (1-\varepsilon)^{0.5}a\left(1+\mathrm{O}(J^{-1})\right). \tag{70}$$

By (43) and Lemma 4(a) we have

$$\sum_{r=1}^{t-1}(t-r)\{2rI\}' = \sum_{r=1}^{t-1}(t-r)\langle 2rI\rangle_{\rho_0} = \sum_{r=1}^{t-1}(t-r)\frac{2rI(2rI+\rho_0)}{2\rho_0} + e_2(r),$$

where $0 \leqslant e_2(r) \leqslant \rho_0/8$. Hence

$$\frac{1}{t(\iota+1)}\sum_{r=1}^{t-1}(t-r)\{2rI\}' =$$

$$\frac{1}{6\rho_0}I^2t(t-1)+\frac{1}{6}I(t-1)+\frac{1}{t(t+1)}\sum_{r=1}^{t-1}re_2(t-r). \tag{71}$$

By (46), (59), and (71) we get

$$\frac{1}{C}(\{U_t\}'+\sum_{r=1}^{t-1}(t-r)\{2rI\}'+C\{V_1\})$$
$$=\frac{I^2}{(q!)^2\rho}(J^{2q}+\mathrm{O}(J^{2q-1}))+g_3(t)(1+\mathrm{O}(J^{-1}))+h_3(t), \tag{72}$$

where

$$g_3(t)=-\frac{I^2J^q}{\rho q!}\left(t+\frac{a^2}{t}\right)+\frac{I^2}{4\rho}\left(t(t+1)+\frac{a^4}{t(t+1)}\right), \tag{73}$$

$$h_3(t)=\frac{2(e_1(t)+\sum_{r=1}^{t-1}re_2(t-r))}{t(t+1)}+e_3.$$

Since

$$\sum_{r=1}^{t}re_2(t+1-r))-\sum_{r=1}^{t-1}re_2(t-r)=\sum_{r=1}^{t}e_2(r),$$

we have

$$h_3(t+1)-h_3(t)=\mathrm{O}(\rho_0/t).$$

Hence, similarly to (67) and by (70) we have

$$(g_3(t+1)-g_3(t))(1+\mathrm{O}(J^{-1}))$$
$$+h_2(t+1)-h_2(t)\begin{cases} >0 & \text{if } t\sim ca, c<1,\\ =L>0 & \text{if } t=\mathrm{o}(a),\end{cases} \tag{74}$$

where

$$L=\frac{I^2a^2}{\rho t^2}\left(\frac{J^q}{q!}-\frac{a^2}{2t}\right)+\mathrm{O}(1).$$

By (55), (56), (70), and Lemma 10 we have

$$\frac{1}{C}F(\bar{\alpha}^*)=\frac{1}{C}\max_{2\leqslant k_0\leqslant t}\left(\frac{t-1}{k_0-1}\sum_{k=2}^{k_0}f_k,0\right)=\mathrm{O}(I^2J^{2q-1.5}). \tag{75}$$

On the other hand, by (61)–(63) we have

$$\frac{1}{C}\left(\{U_t\}'+\sum_{r=1}^{t-1}(t-r)\{2rI\}'+C\{V_1\}+F(\bar{\alpha}^*)\right)$$
$$=\begin{cases}\frac{I^2}{(q!)^2\rho}(J^{2q}-2qJ^{2q-0.5}+\mathrm{O}(J^{2q-1})) & \text{if } t=t^*,\\ \frac{I^2}{(q!)^2\rho}(J^{2q}-\frac{2-\varepsilon}{(1-\varepsilon)^{0.5}}qJ^{2q-0.5}+\mathrm{O}(J^{2q-1})) & \text{if } V_t=(1-\varepsilon)b_{q-1}.\end{cases} \tag{76}$$

Let $U_t \leqslant b_{q-1}$. Then $s \leqslant q-2$ in (60). By (75) and (71) and similarly to (58) and (59) we have

$$\frac{1}{C}\left(\{U_t\}'+\sum_{r=1}^{t-1}(t-r)\{2rI\}'+C\{V_1\}+F(\bar{\alpha}^*)\right)=\mathrm{O}(I^2J^{2q-1}). \tag{77}$$

By (61), (69), (72)–(77), Lemma 11(a), (c), and Theorem 1 we get Theorem 4 for $q>2$.

(4) Finally, let $q=2$. By (53) we have

$$\frac{t-1}{k_0}\sum_{k=2}^{k_0+1}f_k-\frac{t-1}{k_0-1}\sum_{k=2}^{k_0}f_k>0 \tag{78}$$

if $V_{k_0}>b_1$. Hence, by (78), Theorem 3, and Lemma 10 we have

$$F(\bar{\alpha}^*)=\begin{cases}D & \text{if } t\leqslant t',\\ \sum_{k=2}^{t}f_k & \text{if } t>t',\end{cases} \tag{79}$$

where

$$t'\geqslant t^*,\quad \sum_{k=2}^{t'}f_k>D\geqslant\sum_{k=2}^{t'-1}f_k.$$

If $F(\bar{\alpha}^*)=\sum_{k=2}^{t}f_k$, then (32) is true for $q=2$. If $F(\bar{\alpha}^*)=D$ and $U_t>b_1$, then

$$\frac{1}{C}F(\bar{\alpha}^*)=\frac{1}{C}D=\mathrm{O}(I^2J/t),\quad t>J(1+\mathrm{O}(J^{-0.5})).$$

Note that

$$\frac{d(I^2J/t)}{dt}=\mathrm{O}(I^2/J)\ll I^2J^{0.5}.$$

Hence, similarly to (69) and (74) and by (71), (75)–(77), and Theorem 1 we get Theorem 4 for $q=2$. □

Remark 1. If $I=1, \rho=1, q\geqslant 1$, and $h=2q$, then $M(q,1,1,J)=N_h(1,J)$ where $N_h(1,J)$ is the smallest maximal element of a DS. Theorems 2 and 4 give better lower bounds on $N_h(1,J)$ for $h>4$ than the lower bounds given in [4,6].

Remark 2. If $I = 1, p \geqslant 1, q \geqslant 1$, and $h = 2q$, then Theorems 1–4 give a generalization of the lower bounds for DS given in [4,6], namely lower bounds on multiple DS.

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